

A Most General Edge Elimination Graph Polynomial

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Abstract

We look for graph polynomials which satisfy recurrence relations on three kinds of edge elimination: edge deletion, edge contraction and deletion of edges together with their end points. Like in the case of deletion and contraction only (W. Tutte, 1954), it turns out that there is a most general polynomial satisfying such recurrence relations, which we call $\xi(G, x, y, z)$. We show that the new polynomial simultaneously generalizes the Tutte polynomial, the matching polynomial, and the recent generalization of the chromatic polynomial proposed by K.Dohmen, A.Pönitz and P.Tittman (2003), including also the independent set polynomial of I. Gutman and F. Harary, (1983) and the vertex-cover polynomial of F.M. Dong, M.D. Hendy, K.T. Teo and C.H.C. Little (2002). We establish two definitions of the new polynomial: first, the most general confluent recursive definition, and then an explicit one, using a set expansion formula, and prove their identity. We further expand this result to edge-labeled graphs as was done for the Tutte polynomial by T. Zaslavsky (1992) and B. Bollobás and O. Riordan (1999). The edge labeled polynomial $\xi_{lab}(G, x, y, z, \bar{t})$ also generalizes the chain polynomial of R.C. Read and E.G. Whitehead Jr. (1999). Finally, we discuss the complexity of computing $\xi(G, x, y, z)$.

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1 Introduction

There are several well-studied graph polynomials, among them the chromatic polynomial, [Big93, GR01, DKT05], different versions of the Tutte polynomial, [Bol99, BR99, Sok05], and of the matching polynomial, [HL72, LP86, GR01], which are known to satisfy certain linear recurrence relations with respect to *deletion* of an edge, *contraction* of an edge, or deletion of an edge together with its endpoints, which we call *extraction* of an edge. The generalization of the chromatic polynomial, which was introduced by K.Dohmen, A.Pönitz and P.Tittman in [DPT03], happens to satisfy such recurrence relation as well. The question that arises is, what is the most general graph polynomial that satisfies similar linear recurrence relation.

In this paper all the graphs are unlabeled unless it is explicitly mentioned; multiple edges and self loops are allowed. We denote by $G = (V, E)$ the graph with vertex set V and edge set E .

1.1 Recursive definition of graph polynomials

We define three basic edge elimination operations on multigraphs:

- *Deletion.* We denote by G_{-e} the graph obtained from G by simply removing the edge e .
- *Contraction.* We denote by $G_{/e}$ the graph obtained from G by unifying the endpoints of e . Note that this operation can cause production of multiple edges and self loops.
- *Extraction.* We denote by $G_{\dagger e}$ the graph induced by $V \setminus \{u, v\}$ provided $e = \{u, v\}$. Note that this operation removes also all the edges adjacent to e .

Additionally, we require the polynomial to be *multiplicative* for disjoint unions, i.e., if $G_1 \oplus G_2$ denotes disjoint union of two graphs, then the polynomial $P(G_1 \oplus G_2) = P(G_1) \cdot P(G_2)$. This is justified by the fact that the polynomials occurring in the literature are usually multiplicative. The initial conditions are defined for an *empty set* (graph without vertices, usually, $P(\emptyset) = 1$) and for a single point $P(E_1)$. With respect to these operations, we recall the known recursive definitions of graph polynomials:

Matching polynomial. There are different versions of the matching polynomial discussed in the literature, for example *matching generating polynomial* $g(G, \lambda) = \sum_{i=0}^n a_i \lambda^i$ and *matching defect polynomial* $\mu(G, \lambda) = \sum_{i=0}^n (-1)^i a_i \lambda^{n-2i}$, where $n = |V|$ and a_i is the number of i -matchings in G . We shall use the bivariate version that incorporates the both above:

$$M(G, x, y) = \sum_{i=0}^n a_i x^{n-2i} y^i \quad (1)$$

The recursive definition of this polynomial is as follows:

$$\begin{aligned}
M(G) &= M(G_{-e}) + y \cdot M(G_{\dagger e}) \\
M(G_1 \oplus G_2) &= M(G_1) \cdot M(G_2) \\
M(E_1) &= x; \\
M(\emptyset) &= 1;
\end{aligned} \tag{2}$$

Tutte polynomial. We recall the definition of classical two-variable Tutte polynomial (cf. for example B.Bollobás [Bol99]):

Definition 1. Let $G = (V, E)$ be a (multi-)graph. Let $A \subseteq E$ be a subset of edges. We denote by $k(A)$ the number of connected components in the spanning subgraph (V, A) . Then two-variable Tutte polynomial is defined as follows

$$T(G, x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{|A|+k(A)-|V|} \tag{3}$$

This polynomial has linear recurrence relation with respect to the operations above:

$$\begin{aligned}
T(G, x, y) &= \begin{cases} x \cdot T(G_{/e}, x, y) & \text{if } e \text{ is a bridge,} \\ y \cdot T(G_{-e}, x, y) & \text{if } e \text{ is a loop,} \\ T(G_{/e}, x, y) + T(G_{-e}, x, y) & \text{otherwise} \end{cases} \\
T(G_1 \oplus G_2, x, y) &= T(G_1, x, y) \cdot T(G_2, x, y) \\
T(E_1) &= 1; \\
T(\emptyset) &= 1;
\end{aligned} \tag{4}$$

However, we shall use in this paper the version of the Tutte polynomial used by A.Sokal [Sok05]:

$$Z(G, q, v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|} \tag{5}$$

The bivariate Sokal polynomial is co-reducible to the Tutte polynomial via

$$T(G, x, y) = (x-1)^{-k(E)} (y-1)^{-|V|} Z(G, (x-1)(y-1), y-1) \tag{6}$$

and has much recurrence relation which does not distinguish whether the edge e is a loop, a bridge, or none of the two:

$$\begin{aligned}
Z(G, q, v) &= v \cdot Z(G_{/e}, q, v) + Z(G_{-e}, q, v) \\
Z(G_1 \oplus G_2, q, v) &= Z(G_1, q, v) \cdot Z(G_2, q, v) \\
Z(E_1) &= q; \\
Z(\emptyset) &= 1;
\end{aligned} \tag{7}$$

Bivariate chromatic polynomial K.Dohmen, A.Pönitz and P.Tittman in [DPT03] introduced a polynomial $P(G, x, y)$ by splitting the available colors into colors for proper and colors for arbitrary colorings.

We prove in this paper that this polynomial satisfies the following recurrence relation:

$$\begin{aligned} P(G, x, y) &= P(G_{-e}, x, y) - P(G_{/e}, x, y) + (x - y) \cdot P(G_{\dagger e}, x, y) \\ P(G_1 \oplus G_2, x, y) &= P(G_1, x, y) \cdot P(G_2, x, y) \\ P(E_1) &= x; \\ P(\emptyset) &= 1; \end{aligned} \tag{8}$$

1.2 A most general edge elimination polynomial

We define the most general confluent linear recurrence relation¹, which can be obtained on unlabeled graphs by introducing new variables, and which does not distinguish between local properties of the edge e which is to be eliminated². We start with the recurrence relation

$$\begin{aligned} \xi(G) &= w \cdot \xi(G_{-e}) + y \cdot \xi(G_{/e}) + z \cdot \xi(G_{\dagger e}) \\ \xi(G_1 \oplus G_2) &= \xi(G_1) \cdot \xi(G_2) \\ \xi(E_1) &= x; \\ \xi(\emptyset) &= 1; \end{aligned} \tag{9}$$

We prove:

Theorem 1. *The recurrence relation (9) is confluent if and only if one of the following conditions are satisfied:*

$$z = 0 \tag{10}$$

$$w = 1 \tag{11}$$

Under the confluence condition (11), which allows more general graph polynomial to be obtained, the recurrence relation (9) is restricted to

$$\begin{aligned} \xi(G, x, y, z) &= \xi(G_{-e}, x, y, z) + y \cdot \xi(G_{/e}, x, y, z) + z \cdot \xi(G_{\dagger e}, x, y, z) \\ \xi(G_1 \oplus G_2, x, y, z) &= \xi(G_1, x, y, z) \cdot \xi(G_2, x, y, z) \\ \xi(E_1, x, y, z) &= x; \\ \xi(\emptyset, x, y, z) &= 1; \end{aligned} \tag{12}$$

From this theorem one sees immediately that the polynomial $\xi(G, x, y, z)$ gives, by choosing appropriate values for the variables and simple prefactors,

¹ The first paper to study general conditions under which linear recurrence relations define a graph invariant is D.N. Yetter [Yet90].

² It is conceivable that recurrence relations with various case distinctions depending on local properties of e and more variables give other “most general” polynomials. This is the reason why we speak of “a most general” edge elimination polynomial in the title of the paper.

the bivariate Sokal polynomial, the bivariate matching polynomial and the bivariate chromatic polynomial with all their respective substitution instances, including the classical chromatic polynomial, the Tutte polynomial, the vertex-cover and the independent set polynomial, [DHTL02, GH83]. The latter two polynomials are already substitution instances of the bivariate chromatic polynomial $P(G, x, y)$ of [DPT03].

In our next result we give an explicit form of the polynomial $\xi(G, x, y, z)$ using 3-partition expansion³:

Theorem 2. *Let $G = (V, E)$ be a (multi)graph. Then the edge elimination polynomial $\xi(G, x, y, z)$ can be calculated as*

$$\xi(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} \quad (13)$$

where by abuse of notation we use $(A \sqcup B) \subseteq E$ for summation over subsets $A, B \subseteq E$, such that the subsets of vertices $V(A)$ and $V(B)$, covered by respective subset of edges, are disjoint: $V(A) \cap V(B) = \emptyset$; $k(A)$ denotes the number of spanning connected components in (V, A) , and $k_{cov}(B)$ denotes the number of covered connected components, i.e. the connected components of $(V(B), B)$.

Remark 3. From Theorem 2 one can see that $\xi(G, x, y, z)$ is a polynomial definable in Monadic Second Order Logic, with quantification over sets of edges ($MSOL_2$), where an order over vertices is to be used for stating "number of connected sets", but the final result is order-independent. We shall not use logic in the sequel of the paper. For details the reader is referred to [Mak05].

1.3 Comparison with the weighted graph polynomial

The weighted graph polynomial $U(G, \bar{x}, y)$ introduced by S.D. Noble and D.J.A. Welsh in [NW99] is defined for a graph $G = (V, E)$ as

$$U(G, \bar{x}, y) = \sum_{A \subseteq E} \prod_{i=1}^{|V|} x_i^{s(i, A)} y^{|A| - r(A)}$$

where $s(i, A)$ denotes the number of connected components of size i in the spanning subgraph (V, A) , and $r(A) = |V| - k(A)$ is the rank of (V, A) .

The main difference between $U(G, \bar{x}, y)$ and $\xi(G, x, y, z)$ is the number of variables, which grows in the case of U and is fixed in the case of ξ . Furthermore, in the definition of $s(i, A)$ the numeric value of the index of the variable x_i is used. This has as a consequence that one cannot freely rename the variables of U . In ξ , as well as in all graph polynomials definable in $MSOL_2$ in an order invariant way, the variables can be renamed. This allows one to show that $U(G, \bar{x}, y)$ is not an $MSOL_2$ -definable polynomial.

³ A more precise name would be "Pair of two disjoint subsets expansion". We chose the name 3-partition expansion, as any two disjoint subsets induce a partition into three sets.

$U(G, \bar{x}, y)$ also gives the Tutte polynomial and the matching polynomial as its substitution instances. One can see that the polynomial $U(G, \bar{x}, y)$ distinguishes between graphs for which $\xi(G, x, y, z)$ gives the same value. As an example we look at the trees shown on Fig. 1. We do not know whether $\xi(G, x, y, z)$ can be obtained as a substitution instance of $U(G, \bar{x}, y)$.

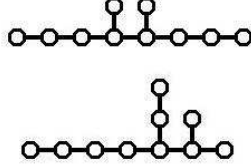


Fig. 1: Non-isomorphic trees having the same $\xi(G, x, y, z)$.

1.4 A labeled version of ξ

For edge-labeled⁴ graphs we define the labeled version of our polynomial: Let $G = (V, E, c)$ be an edge-labeled multigraph s.t. $c : E \mapsto \Lambda$, where Λ is a set of labels, without any algebraic structure defined over it, and suppose that for each $\lambda \in \Lambda$ three elements w_λ , y_λ and z_λ of a field are chosen. Then using the same approach as for the unlabeled version, we define a linear recurrence relation:

$$\begin{aligned} \xi_{lab}(G) &= w_{c(e)} \cdot \xi_{lab}(G_{-e}) + y_{c(e)} \cdot \xi_{lab}(G_{/e}) + z_{c(e)} \cdot \xi_{lab}(G_{\dagger e}) \\ \xi_{lab}(G_1 \oplus G_2) &= \xi_{lab}(G_1) \cdot \xi_{lab}(G_2) \\ \xi_{lab}(E_1) &= x; \\ \xi_{lab}(\emptyset) &= 1; \end{aligned} \tag{14}$$

Note that we do not introduce weights on the vertices, as this would make the definition of the edge contraction unclear, unless we define an algebraic structure over Λ . For example, if Λ was a ring, we could define the label of the vertex produced by a contraction of an edge $\{u, v\}$ to be the sum of the labels of u and v . In that case, we would get a generalization of the weighted graph polynomial for labeled graphs $W(G, \bar{x}, y)$, also introduced by S.D.Noble and D.J.A.Welsh in [NW99]. However, this polynomial is "too strong", in sense that it has the same definability problems as $U(G, \bar{x}, y)$ discussed in Section 1.3.

For the labeled case we prove

Theorem 4. *Every one of the conditions*

$$\forall e \in E (z_{c(e)} = 0) \tag{15}$$

$$\forall e \in E (w_{c(e)} = 1) \wedge \forall e_1, e_2 \in E (y_{c(e_1)} z_{c(e_2)} = y_{c(e_2)} z_{c(e_1)}) \tag{16}$$

is sufficient for the recurrence relation (14) being confluent.

⁴ In [BR99] they speak of edge-colorings rather than edge-labelings. As we also discuss chromatic polynomials we prefer our terminology as it avoids confusions.

Remark 5. The conditions in Theorem 4 are not necessary. To see this we look at a graph with two connected components and use condition (15) for edges in the first component and condition (16) for edges in the second component.

Under the confluence condition (16), which allows more general graph polynomial to be obtained, the recurrence relation (14) is restricted to

$$\begin{aligned}\xi_{lab}(G) &= \xi_{lab}(G_{-e}) + y \cdot t_{c(e)} \cdot \xi_{lab}(G_{/e}) + z \cdot t_{c(e)} \cdot \xi_{lab}(G_{\dagger e}) \\ \xi_{lab}(G_1 \oplus G_2) &= \xi_{lab}(G_1) \cdot \xi_{lab}(G_2) \\ \xi_{lab}(E_1) &= x; \\ \xi_{lab}(\emptyset) &= 1;\end{aligned}\tag{17}$$

where

$$\begin{aligned}y_{c(e)} &= y \cdot t_{c(e)} \\ z_{c(e)} &= z \cdot t_{c(e)}\end{aligned}\tag{18}$$

x , y and z are unlabeled variables, and \bar{t} is the unique solution of (18). Like the unlabeled case, we also introduce the explicit form:

Theorem 6. *The expression*

$$\xi_{lab}(G, x, y, z, \bar{t}) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B)} \left(\prod_{e \in A \sqcup B} (y t_{c(e)}) \right) \left(\frac{z}{xy} \right)^{k_{cov}(B)} \tag{19}$$

defines the same graph polynomial as the recurrence relation (17).

Note that the degree of x and y in the denominator does never exceed the degree of the respective variable in the nominator.

Remark 7. The *labeled Sokal polynomial* [Sok05], *Zaslavsky's normal function of the colored matroid* [Zas92], *Heilmann and Lieb's labeled matching polynomial* [HL72], and the *chain polynomial* [RJ99, Tra02] are substitution instances of $\xi_{lab}(G, x, y, z, \bar{t})$ up to a simple prefactor.

The remainder of the paper is organized as follows: in Section 2 we prove the recurrence relation of the generalized chromatic polynomial. In Section 3 we establish the most general linear recurrence relation with respect to the three edge elimination operations, restrict it to be multiplicative and confluent, and then prove the confluence property of the resulting function. In Section 4 we establish the explicit function as in Theorem 2, and prove that it defines the same polynomial. The section 5 expands our results to the edge-labeled graphs. The section 6 contains examples of known graph polynomials which can be obtained as substitution instances of the edge elimination polynomial. Finally, in Section 7 we deal with the complexity of its computation.

2 The recursive definition of the generalized chromatic polynomial

Recall the definition given by K.Dohmen, A.Pönitz and P.Tittman in [DPT03]: There are two disjoint sets of colors Y and Z ; a generalized coloring of a graph $G = (V, E)$ is a map $\phi : V \mapsto (Y \sqcup Z)$ such that for all $\{u, v\} \in E$, if $\phi(u) \in Y$ and $\phi(v) \in Y$, then $\phi(u) \neq \phi(v)$ (The set Y is called therefore "the proper colors"). For two positive integers $x > y$, the value of the polynomial is the number of generalized colorings of G by x colors, y of them are proper. We enhance this definition to multigraphs by the following:

- (i) A self-loop can be colored only by a color in $X \setminus Y$;
- (ii) A multiple edge does not affect colorings.

Let $G = (V, E)$ be a graph, and $P(G, x, y)$ be the number of generalized colorings defined above. Let $v \in V$ be any vertex. We denote by $P^v(G, x, y)$ the number of generalized colorings of G , when v is not colored by a proper color, i.e. $\phi(v) \in X \setminus Y$.

Proposition 8. $P^v(G, x, y) = (x - y) \cdot P(G_{-v}, x, y)$, where G_{-v} denotes the subgraph of G induced by $V \setminus \{v\}$.

Proof. By inspection: the vertex v can have any color in $X \setminus Y$, and the coloring of the remainder does not depend on it. \square

Let $e = \{u, v\} \in E$ be any edge of G , which is not a self-loop and not a multiple edge. Consider the number of colorings of G_{-e} . Any such coloring is either a coloring of G , or a coloring of $G_{/e}$, when the vertex $u = v$, which is produced by the contraction, is colored by a proper color. Together with Proposition 8, that raises:

$$P(G, x, y) = P(G_{-e}, x, y) - P(G_{/e}, x, y) + (x - y) \cdot P(G_{\dagger e}, x, y) \quad (20)$$

One can easily check that this equation is satisfied also for loops and multiple edges. Together with the fact that a singleton can be colored by any color, and the fact that the number of colorings is multiplicative, this proves the recursive definition (8).

3 The most general recurrence relation

We are looking for the most general linear recurrence relation with respect to edge deletion, edge contraction and edge extraction operation that can be obtained by introducing new variables. Recall that we are interested in a *graph invariant*, e.i. the resulting function should not depend on the order of graph deconstruction. Moreover, this invariant should be a multiplicative graph polynomial.

From this consideration alone we obtain the initial condition and the product rule:

$$\begin{aligned}\xi(G_1 \oplus G_2) &= \xi(G_1) \cdot \xi(G_2) \\ \xi(\emptyset) &= 1;\end{aligned}\tag{21}$$

Indeed, the disjoint union with an empty set gives the same graph, so the resulting function should also remain the same.

At this stage, we formulate the edge elimination rule introducing a new variable wherever we can. We set

$$\begin{aligned}\xi(G, x, y, z, t) &= t \cdot \xi(G_{-e}, x, y, z, t) + y \cdot \xi(G_{/e}, x, y, z, t) + z \cdot \xi(G_{\dagger e}, x, y, z, t) \\ \xi(E_1, x, y, z, t) &= x;\end{aligned}\tag{22}$$

Let G be a graph as presented on Fig. 2. Note that the subgraphs H_1 , H_{1-u} , H_2 and H_{2-w} can be different and have (in general) different ξ .

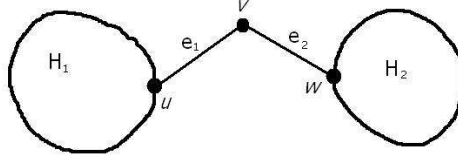


Fig. 2: The graph for testing of the confluence property.

Since we are looking for a graph invariant, we must obtain the same result by applying the edge elimination rule first on the edge e_1 and then on the edge e_2 , as in case when we apply the edge elimination rule first on the edge e_2 and then on the edge e_1 .

$$\begin{aligned}\xi(G) &= t \cdot \xi(G_{-e_1}) + y \cdot \xi(G_{/e_1}) + z \cdot \xi(G_{\dagger e_1}) = \\ &= t \cdot \xi(H_1) \cdot [x \cdot t \cdot \xi(H_2) + y \cdot \xi(H_2) + z \cdot \xi(H_{2-w})] + \\ &\quad y \cdot [t \cdot \xi(H_1) \xi(H_2) + y \cdot \xi(G_{/e_1/e_2}) + z \cdot \xi(H_{1-u}) \xi(H_{2-w})] + \\ &\quad z \cdot \xi(H_{1-u}) \xi(H_2)\end{aligned}\tag{23}$$

On the other hand,

$$\begin{aligned}\xi(G) &= t \cdot \xi(G_{-e_2}) + y \cdot \xi(G_{/e_2}) + z \cdot \xi(G_{\dagger e_2}) = \\ &= t \cdot \xi(H_2) \cdot [x \cdot t \cdot \xi(H_1) + y \cdot \xi(H_1) + z \cdot \xi(H_{1-u})] + \\ &\quad y \cdot [t \cdot \xi(H_1) \xi(H_2) + y \cdot \xi(G_{/e_1/e_2}) + z \cdot \xi(H_{1-u}) \xi(H_{2-w})] + \\ &\quad z \cdot \xi(H_{2-w}) \xi(H_1)\end{aligned}\tag{24}$$

Hence, we have either $z = 0$ or $t = 1$ as a necessary condition of the confluence. In case of $z = 0$ the resulting function is a substitution instance of the Sokal polynomial:

$$\xi(G, x, y, 0, t) = t^{|E|} \cdot Z(G, x, \frac{y}{t})\tag{25}$$

We leave the proof of that fact to the reader. Since the Sokal polynomial can be also obtained when $t = 1$, the latter case is considered more general, and it will be further investigated. That brings us back to the recurrence relation (12). To complete the proof of Theorem 1, we need now to show that *the recurrence relation (12) is confluent*.

It is enough to prove that any two steps of the graph decomposition using (12) are interchangeable. This includes two parts:

- Decomposition of a graph by elimination of any two edges in different order;
- Edge elimination and disjoint union.

The proof of both parts is rather technical and left to the reader.

4 The explicit form or the polynomial $\xi(G, x, y, z)$

In this section we prove Theorem 2. In order to do so, we need to show that

- The expression (13) satisfies the initial conditions of (12);
- The expression (13) is multiplicative;
- The expression (13) satisfies the edge elimination rule of (12).

Then by induction on the number of edges in G the theorem holds. The first fact is trivial; the second one can be easily checked by reader. Indeed, the summation over subsets of edges of $G(V, E) = G_1(V_1, E_1) \oplus G_2(V_2, E_2)$ can be regarded as a summation over the subsets of E_1 , and then independently over the subsets of E_2 . Therefore, we just need to prove that

Lemma 9. *The explicit expression given by (13) satisfies the edge elimination rule of (12).*

Proof. Let $G = (V, E)$ be the (multi)graph of interest. Let $N(G)$ be defined as

$$N(G, x, y, z) = \sum_{(A \sqcup B) \subseteq E} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} \quad (26)$$

where $k(A)$ denotes the number of connected components in (V, A) , and $k_{cov}(B)$ denotes the number of the connected components of $(V(B), B)$, where $V(B) \subseteq V$ are the vertices covered by the edges of B . Let e be the edge we have chosen to reduce. Any particular choice of A and B can be regarded as a vertex-disjoint edge coloring in 2 colors A and B, when part of the edges remains uncolored. We divide all the coloring into three disjoint cases:

- Case 1: e is uncolored;

- Case 2: e is colored by B , and it is the only edge of a colored connected component;
- Case 3: All the rest. That means, e is colored by A , or e is colored by B but it is not the only edge of a colored connected component.

In the case 1, we just sum over colorings of G_{-e} :

$$N_1(G) = \sum_{(A \sqcup B) \models \text{Case 1}} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} = N(G_{-e}) \quad (27)$$

In the case 2, the edge e is a connected component of $(V(B), B)$. Therefore, if we analyze now $N(G_{\dagger e})$, we will get

- The number of edges colored by A is the same;
- The number of edges colored by B is reduced by one;
- The total number of colored connected components is reduced by one;
- The number of covered connected components colored B is reduced by one;

This gives us

$$N_2(G) = \sum_{(A \sqcup B) \models \text{Case 2}} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} = z \cdot N(G_{\dagger e}) \quad (28)$$

And finally, in the case 3, e is a part of a bigger colored connected component, or it is alone a connected component colored by A . In this case, we analyze the colorings of $G_{/e}$:

- Either $|A|$ or $|B|$ is reduced by 1, the other remained the same;
- The total number of colored connected components remained the same;
- The number of covered connected components colored B remained the same.

According to the above,

$$N_3(G) = \sum_{(A \sqcup B) \models \text{Case 3}} x^{k(A \sqcup B) - k_{cov}(B)} \cdot y^{|A| + |B| - k_{cov}(B)} \cdot z^{k_{cov}(B)} = y \cdot N(G_{/e}) \quad (29)$$

which together with $N(G) = N_1(G) + N_2(G) + N_3(G)$ completes the proof. \square

5 The edge elimination polynomial of a labeled graph

To obtain the edge-labeled version of our polynomial, we use the same approach as in Section 3: we are looking for a multiplicative graph invariant satisfying linear recurrence relation with respect to the edge elimination operations. We start with

$$\begin{aligned}\xi_{lab}(G_1 \oplus G_2) &= \xi_{lab}(G_1) \cdot \xi_{lab}(G_2) \\ \xi_{lab}(\emptyset) &= 1;\end{aligned}\tag{30}$$

and define an edge elimination rule introducing a new variable wherever we can (now every variable has index e for the edge which is currently being eliminated⁵).

$$\begin{aligned}\xi_{lab}(G) &= w_e \cdot \xi(G_{-e}) + y_e \cdot \xi_{lab}(G_{/e}) + z_e \cdot \xi_{lab}(G_{\dagger e}) \\ \xi_{lab}(E_1) &= x;\end{aligned}\tag{31}$$

The same considerations as in Section 3, using the same graph (Fig. 1), we get that the recursion (31) is confluent when either $z_e = 0$ or $w_e = 1$ and $y_{e_1}z_{e_2} = y_{e_2}z_{e_1}$. One can expand this result to any two edges of a connected component. Since the graph in general can be connected, and our recurrence relation should be confluent for *every* graph, we get the following restrictions:

- $z_e = 0$ for *every* edge e , or
- $w_e = 1$, $y_e = y \cdot t_e$ and $z_e = z \cdot t_e$ for *every* edge e (here y and z do not depend on e).

In the first case we obtain an instance of the labeled Sokal polynomial:

$$\xi_{lab}(G, \bar{w}, x, \bar{y}, \bar{0}) = \left(\prod_{e \in E} w_e \right) \cdot Z(G, q, \bar{v})\tag{32}$$

where $q = x$ and $v_e = \frac{y_e}{w_e}$. Since the Sokal polynomial can be also obtained when $\bar{w} = \bar{1}$, the latter case is considered more general. That brings us to the recurrence relation (17). We have now to prove two propositions:

Proposition 10. *The recurrence relation (17) is confluent.*

Proposition 11. *The formula (19) defines the same polynomial as the recurrence relation (17).*

Both the proofs are similar to the respective unlabeled version and left to the reader.

⁵We do not use an index for vertices, because the vertex set of the graph is being changed during decomposition. For this reason, we cannot call this polynomial "the most general"

6 Application to some known graph polynomials

In this section we present different known graph polynomials as substitution instances of $\xi(G)$ and $\xi_{lab}(G)$. Two issues should be addressed here:

Zero coefficients: When some of the arguments x , y or z of our polynomial is zero, we generally get 0 in all the summands that contain this variable in some positive power, and an uncertainty of kind 0^0 in all the summands that contain it in power 0. However, as of being a polynomial, our function is continuous, and thus we can use the fact that for any nonnegative integer k ,

$$x^k|_{x=0} = \lim_{x \rightarrow 0} x^k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence, if in our substitution some variable turns 0, the value of the resulting polynomial is still well-defined.

Multiple edges and loops: Some of the graph polynomials are defined only for simple and loop-free graphs. However, their definition can be easily generalized to multigraphs, such that the equality holds in case of a simple input graph.

Labeled versions of the Tutte polynomial

Proposition 12. *The Sokal polynomial (in both unlabeled and labeled versions) can be obtained by*

$$\begin{aligned} Z(G, q, v) &= \xi(G, q, v, 0) \\ Z_{lab}(G, q, \bar{v}) &= \xi_{lab}(G, q, 1, 0, \bar{v}) \end{aligned}$$

in particular, the chromatic polynomial can be obtained by

$$\chi(G, \lambda) = Z(G, \lambda, -1) = \xi(G, \lambda, -1, 0)$$

Proof. By inspection of summands with $B = \emptyset$. All the other summands are eliminated by $z = 0$. \square

By a simple substitution of variables, we get the following three corollaries:

Corollary 13. *The classical Tutte polynomial can be obtained by*

$$T(G, x, y) = (x - 1)^{-k(E)} \cdot (y - 1)^{-|V|} \cdot \xi(G, (x - 1)(y - 1), (y - 1), 0)$$

Recall that $r(S) = |V| - k(S)$ is the rank of the spanning subgraph with edge set S . The Zaslavsky's normal function of the colored matroid, applied to a graph $G = (V, E)$ with edge coloring function $c : E \mapsto \Lambda$, is defined by

$$R(G, c) = \sum_{S \subseteq E} \left(\prod_{e \in S} x_{c(e)} \right) \left(\prod_{e \notin S} y_{c(e)} \right) (x - 1)^{r(E) - r(S)} (y - 1)^{|S| - r(S)}.$$

Corollary 14. *The Zaslavsky's normal function of the edge-colored graph can be obtained by*

$$R(G, c) = (x')^{-k(E)} \cdot (y')^{-|V|} \cdot \left(\prod_{e \in E} y_{c(e)} \right) \cdot \xi_{lab}(G, x' y', y', 0, \bar{t})$$

where $t_{c(e)} = \frac{x_{c(e)}}{y_{c(e)}}$ and $x' = x - 1, y' = y - 1$.

The chain polynomial $Ch(G, \omega, \bar{u})$ was first introduced in [RJ99] and can also be defined, cf. [Tra02], as

$$Ch(G, \omega, \bar{u}) = \sum_{S \subseteq E} (1 - \omega)^{|S| - r(S)} \prod_{e \in E - S} u_e.$$

From [Tra02] we get

Corollary 15. *The chain polynomial can be obtained by*

$$Ch(G, \omega, \bar{u}) = \left(\prod_{e \in E} u_e \right) \cdot (1 - \omega)^{-|V|} \cdot \xi_{lab}(G, 1 - \omega, 1, 0, \bar{v})$$

where $v_e = \frac{1 - \omega}{u_e}$.

Matching polynomials

The next two propositions deal with various forms of matching polynomials:

Proposition 16. *The generalized matching polynomial (1) can be obtained by*

$$M(G, x, y) = \sum_{i=0}^n a_i x^{n-2i} y^i = \xi(G, x, 0, y)$$

In particular, the generating matching polynomial is $g(G, x) = \xi(G, 1, 0, x)$ and the defect matching polynomial is $\mu(G, x) = \xi(G, x, 0, -1)$

Proposition 17. *The original Heilmann and Lieb's multivariate matching polynomial introduced in [HL72] can be obtained by*

$$M_{col}(G, \bar{x}, \bar{y}) = \sum_{\substack{M \subseteq E, \\ M \text{ is a matching}}} \prod_{e=\{u,v\} \in M} y_e x_u x_v = \xi_{lab}(G, 1, 0, 1, \bar{t}) \quad (33)$$

where $t_e = y_e x_u x_v$ for every edge $e = \{u, v\}$.

Proof. By inspection of non-zero summands of $\xi(G)$. There should be no edges in A , and every edge of B should be in different connected component, so B has to be a matching. \square

Finally, the Dohmen-Pönitz-Tittman generalization of the chromatic polynomial [DPT03] is also a substitution instance of $\xi(G)$:

Proposition 18.

$$P(G, x, y) = \xi(G, x, -1, x - y)$$

Proof. Using recursion scheme (8), by induction on number of edges $|E|$. \square

Using results of K.Dohmen, A.Pönitz and P.Tittman [DPT03] we can also derive that the independence polynomial (which is a substitution instance of $P(G, x, y)$) is also a substitution instance of $\xi(G)$.

7 Computational complexity of $\xi(G)$

In this section we analyze the complexity of computation of $\xi(G)$ and $\xi_{lab}(G)$. In general, these polynomials are $\#\mathbf{P}$ -hard to compute, as every instance stated in the previous section is $\#\mathbf{P}$ -hard. Recall that, according to Remark 3, the formulas (13) and (19) can be used to give an order invariant definition in Monadic Second Order Logic, with quantification over sets of edges, and an auxiliary order.

Hence, due to the general theorem from [Mak05, Mak04], we have

Proposition 19. $\xi(G)$ and $\xi_{lab}(G)$ are polynomial time computable on graphs of tree-width at most k where the exponent of the run time is independent of k .

Recall also from Remark 3 that the weighted graph polynomial $U(G, \bar{x}, y)$ is not definable using *MSOL*, and, hence, the results of [Mak05, Mak04] are not applicable. Indeed, the run time of the algorithm introduced by C.Noble in [Nob08] for graphs of tree width at most k is polynomial, but its highest degree depends on k .

The drawback of the general method of [Mak05, Mak04] lies in the huge hidden constants, which make it practically unusable. However, an explicit dynamic algorithm for computing the polynomial $\xi_{lab}(G)$ on graphs of bounded tree-width, given the tree decomposition of the graph, where the constants are simply exponential in k , can be constructed along the same ideas as presented in [Tra06, FMR08].

8 Open questions

Difficult point property: In general, the computation of $\xi(G)$ is $\#\mathbf{P}$ -hard. However, we know that for some x, y and z , it can be easy. F.Jaeger, D.Vertigan and D.Welsh define in [JWV90] the set of the points of the (x, y) -plane in which the computation of the induced graph invariants of the bivariate Tutte polynomial is easy, proving that the remaining points are $\#\mathbf{P}$ -hard. Similar theorems have been proven for the interlace polynomial, the cover polynomial and the colored Tutte polynomial [BD07, BH07, BDM07]. In all the cases the “easy” points lay in a semi-algebraic subset of the polynomial domain of lower dimension.

Question 1. *Describe the set of points for which the induced graph invariants $\xi(G)$ and $\xi_{lab}(G)$ are easy to compute.*

Distinctive power: We know that the polynomial $\xi(G)$ has at least the same distinctive power as the Tutte polynomial and the bivariate chromatic polynomial together, but more than every one of them individually. Indeed, since $T(G, x, y)$ and $P(G, x, y)$ are both substitution instances of $\xi(G)$, if $\xi(G)$ coincides for two graphs, so do $T(G, x, y)$ and $P(G, x, y)$. On the other hand, we do not know whether $\xi(G)$ has more distinctive power.

Question 2. *Are there two graphs G_1, G_2 such that for all x, y we have*

$$T(G_1, x, y) = T(G_2, x, y) \text{ and } P(G_1, x, y) = P(G_2, x, y)$$

but such that for some x, y, z

$$\xi(G, x, y, z) \neq \xi(G_2, x, y, z)?$$

Complexity on graphs of bounded clique-width: We have seen in Proposition 19 that for graphs of tree-width at most k computing the edge reduction polynomials $\xi(G)$ and $\xi_{lab}(G)$ is fixed parameter tractable (FPT) in the sense of [DF99, FG06]. Another graph parameter, introduced in [CO00] and discussed there is the clique-width. It is open whether the Tutte polynomial is fixed parameter tractable for graphs of clique-width at most k , [GHN05, MRAG06].

Question 3. *Are the polynomials $\xi(G)$ and $\xi_{lab}(G)$ fixed parameter tractable for graph classes of bounded clique-width?*

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